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Low-frequency modulation of thermal instability

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A Boussinesq fluid is heated from below. The applied temperature gradient is the sum of a steady component and a low-frequency sinusoidal component. An asymptotic solution is obtained which describes the behaviour of infinitesimal disturbances to this configuration. The solution is discussed from the viewpoint of the stability or otherwise of the basic state, and possible stability criteria are analyzed. Some comparison is made with known experimental results.

1. Introduction

In a recent paper, Venezian (1969) has considered the following problem. An incompressible viscous fluid is confined between parallel horizontal planes, and is of infinite horizontal extent. An equilibrium state is maintained under the action of a vertical temperature gradient, which combines a uniform component (the classical thermal instability situation) together with a component varying sinusoidally in time. This equilibrium is unstable to small disturbances at certain values of the relevant parameters, which are to be found.

A similar problem was considered earlier by Gershuni & Zhukhovitskii (1963). In their work, however, the temperature fluctuations obey a rectangular law, instead of being sinusoidal. Other special restrictions are introduced, so that the results are not directly relevant to those of Venezian or of the present paper.

In Venezian's (1969) work the periodic component of the temperature field is of small amplitude compared with the steady component. The object then is to determine the modulating effect of the oscillation on the stability characteristics of the mean gradient. The solution is obtained by an approximation method based on the smallness of the amplitude ratio.

One of the aims of Venezian's paper is to compare his solution with some experimental results obtained by Donnelly (1964). The latter has studied the behaviour of disturbances in circular Couette flow between coaxial cylinders, when the motion of the inner cylinder consists of a small oscillation about a steady rotation, while the outer cylinder is at rest. Donnelly finds that the critical Taylor number is increased in the presence of the periodic motion, the magnitude of the enhancement being a function of the oscillation frequency and amplitude.

In particular, Donnelly's experiments indicate an optimum value of the (dimensionless) frequency at which the modulation is most effective. Venezian's theory, on the other hand, does not yield any such feature, and he finds, in fact, that for moderate Prandtl numbers the modulated critical Rayleigh number increases monotonically with decreasing frequency: the enhancement appears to be a maximum in the limit of zero frequency. In view of the well-known similarity between the thermal and Couette instability problems, a qualitative correspondence between the theory and the observations might have been expected, and the discrepancy is rather surprising.

In explanation it is suggested by Venezian that linear stability theory ceases to be applicable when the frequency is sufficiently small. A similar point was made by Rosenblat (1968) in a discussion of inviscid, time-periodic Couette flow. The basis of this suggestion is the following. At very low frequencies the equilibrium state oscillates slowly about its mean value so that, when the latter is close to critical, the instantaneous value of the Rayleigh number is supercritical during nearly half the cycle. It may therefore happen that disturbances grow to a sufficient size for non-linear effects to become important, in which case their behaviour will not be adequately described by linear theory.

The present paper attempts to achieve two objectives: (i) To find an asymptotic solution of the problem in the case of very small frequency, with arbitrary amplitude ratio. (ii) To interpret this solution in the light of the foregoing remarks. This involves a re-evaluation of the criterion applied to determine the critical Rayleigh number.

2. Perturbation equations

The bounding planes are located at $z = 0$ and $z = d$ with respect to a Cartesian co-ordinate system, and, as Venezian (1969) has done, we shall take them to be *free surfaces*. The governing equations in the Boussinesq approximation are (cf. Chandrasekhar 1961)

$$\nabla . \mathbf{q} = 0, \tag{2.1}$$

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} . \nabla)\mathbf{q} = -\frac{1}{\rho_m}\nabla p + [1 - \alpha(T - T_m)]\mathbf{X} + \nu\nabla^2\mathbf{q}, \tag{2.2}$$

$$\frac{\partial T}{\partial t} + (\mathbf{q} . \nabla)T = \kappa\nabla^2 T, \tag{2.3}$$

where ρ_m, T_m are (constant) averages of density and temperature respectively, $\mathbf{X} = (0, 0, -g)$ is the body force per unit mass, α is the coefficient of volumetric expansion, and the remaining notation is standard.

These equations admit an equilibrium solution in which $\mathbf{q} = 0$, $T = \overline{T}(z,t)$ is a solution of

$$\frac{\partial \overline{T}}{\partial t} = \kappa \frac{\partial^2 \overline{T}}{\partial z^2}, \tag{2.4}$$

and the pressure $\overline{p}(z,t)$ balances the buoyancy force. The precise form of $\overline{T}$ clearly depends on the nature of the applied heating, and in this paper we shall restrict ourselves to the case where the upper surface is maintained at zero temperature, while the lower surface has a temperature which oscillates about a non-zero mean. (Venezian considered other situations in addition to this.) The boundary conditions for (2.4) then are

$$\left.\begin{aligned} \overline{T} &= \beta d[1 + \epsilon\cos\Omega t] \quad &\text{on} \quad z = 0,\\ \overline{T} &= 0 \quad &\text{on} \quad z = d,\end{aligned}\right\} \tag{2.5}$$

β, ϵ being real constants.

For the disturbances, we Fourier-analyze in the xy plane, and substitute into (2.1)–(2.3) the expressions

$$\left.\begin{aligned} \mathbf{q} = \mathbf{q}(z,t)\,e^{i(a_x x + a_y y)}, \quad T = \overline{T} + \theta(z,t)\,e^{i(a_x x + a_y y)}, \\ p = \overline{p} + p(z,t)\,e^{i(a_x x + a_y y)}, \end{aligned}\right\} \tag{2.6}$$

and neglect non-linear terms. After some elimination it emerges that the linearized perturbation equations can be reduced to a single equation for w, the vertical component of velocity. We obtain

$$\left(\frac{\partial^2}{\partial z^2} - a^2\right)\frac{\partial^2 w}{\partial t^2} - (\kappa+\nu)\left(\frac{\partial^2}{\partial z^2} - a^2\right)^2 \frac{\partial w}{\partial t} + \kappa\nu\left(\frac{\partial^2}{\partial z^2} - a^2\right)^3 w - g\alpha a^2 \frac{\partial \overline{T}}{\partial z} w = 0, \tag{2.7}$$

where

$$a = (a_x^2 + a_y^2)^{\frac{1}{2}} \tag{2.8}$$

is the horizontal wave-number.

It is useful to express the quantities involved in (2.7) in dimensionless form. We put

$$\left.\begin{aligned} z' = z/d, \quad a' = ad, \quad t' = \Omega t, \\ \overline{T}' = \overline{T}/\beta d, \quad w' = (\nu/g\alpha a^2 d^4)\,w, \end{aligned}\right\} \tag{2.9}$$

and introduce the parameters

$$\omega = \Omega d^2/\kappa, \quad \sigma = \nu/\kappa, \quad R = \frac{g\alpha\beta d^4}{\kappa\nu}. \tag{2.10}$$

Substitution of (2.9) now gives in place of (2.7) the equation

$$\left(\frac{\partial^2}{\partial z^2} - a^2\right)\frac{\partial^2 w}{\partial t^2} - \frac{(1+\sigma)}{\omega}\left(\frac{\partial^2}{\partial z^2} - a^2\right)^2 \frac{\partial w}{\partial t} + \frac{\sigma}{\omega^2}\left(\frac{\partial^2}{\partial z^2} - a^2\right)^3 w - \frac{\sigma R a^2}{\omega^2}\frac{\partial \overline{T}}{\partial z} w = 0, \tag{2.11}$$

(in which the primes have been omitted). The free surface boundary conditions to be satisfied by w are (Chandrasekhar 1961)

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial^4 w}{\partial z^4} = 0 \quad \text{on} \quad z = 0 \quad \text{and} \quad z = 1. \tag{2.12}$$

The temperature gradient $\partial\overline{T}/\partial z$ in (2.11) is obtained from the dimensionless form of (2.4) and (2.5). It is easily shown that

$$\frac{\partial \overline{T}}{\partial z} = -1 - \epsilon\,\mathrm{Re}\left\{\frac{(i\omega)^{\frac{1}{2}}\cosh\left[(i\omega)^{\frac{1}{2}}(1-z)\right]}{\sinh\left[(i\omega)^{\frac{1}{2}}\right]}e^{it}\right\}. \tag{2.13}$$

The solution w of (2.11) can be represented by a Fourier series in z with time-dependent coefficients. We put

$$w(z,t) = \sum_{m=1}^{\infty} x_m(t)\sin m\pi z, \tag{2.14}$$

each component of which identically satisfies all the boundary conditions (2.12). Now substitute this series into (2.11), multiply through by $\sin n\pi z\ (n = 1, 2, \ldots)$, and integrate with respect to z over the interval $(0, 1)$. The outcome is a system of ordinary differential equations for the functions $x_n(t)$, namely

$$\frac{d^2x_n}{dt^2}+\frac{(1+\sigma)\lambda_n}{\omega}\frac{dx_n}{dt}+\frac{\sigma}{\omega^2\lambda_n}(\lambda_n^3-Ra^2)x_n$$
$$-\frac{\epsilon Ra^2\sigma}{\omega^2\lambda_n}\sum_{m=1}^{\infty}x_m(p_{mn}e^{it}+\tilde{p}_{mn}e^{-it})=0 \quad (n=1,2,\ldots), \quad (2.15)$$

where $\sim$ denotes conjugate complex,

$$\lambda_n = n^2\pi^2+a^2 \quad (2.16)$$

and
$$p_{mn}=\frac{(i\omega)^{\frac{1}{2}}}{\sinh[(i\omega)^{\frac{1}{2}}]}\int_0^1 \sin m\pi z \sin n\pi z \cosh[(i\omega)^{\frac{1}{2}}(1-z)]\,dz$$
$$=\tfrac{1}{2}i\omega\left[\frac{1}{\pi^2(n-m)^2+i\omega}-\frac{1}{\pi^2(n+m)^2+i\omega}\right]. \quad (2.17)$$

3. Low-frequency approximation

So far no approximations have been made in respect of either ϵ or ω, and (2.15) is, apart from nomenclature, identical with that obtained by Venezian (1969). The latter now restricts further developments to the case $\epsilon \ll 1$, and solves (2.15) by an expansion in powers of ϵ.

Instead of this approach, we shall in this paper obtain an approximate solution for $\omega \ll 1$. It is evident that the system (2.15) is singular as $\omega \to 0$, but it is of a classical type amenable to asymptotic solution by standard WKB techniques. This approach does not limit ϵ to small values, although there will be a restriction imposed later.

We first note that the coupling coefficients $p_{mn}(\omega)$ admit expansions in powers of ω, with, in particular, $p_{mn}(0)=\frac{1}{2}\delta_{mn}$. Hence it is easy to show that the system (2.15) can be written

$$\omega^2\frac{d^2x_n}{dt^2}+\omega(1+\sigma)\lambda_n\frac{dx_n}{dt}+\frac{\sigma}{\lambda_n}[\lambda_n^3-Ra^2(1+\epsilon\cos t)]x_n$$
$$-\frac{\epsilon Ra^2\sigma}{\lambda_n}\sum_{m=1}^{\infty}x_m[\omega q_{mn}^{(1)}\sin t+\omega^2 q_{mn}^{(2)}\cos t+O(\omega^3)]=0 \quad (n=1,2,\ldots), \quad (3.1)$$

where
$$q_{mn}^{(1)}=\begin{cases}\dfrac{1}{4n^2\pi^2} & (m=n),\\[2ex] \dfrac{1}{\pi^2(m+n)^2}-\dfrac{1}{\pi^2(m-n)^2} & (m\neq n),\end{cases} \quad (3.2)$$

and
$$q_{mn}^{(2)}=\begin{cases}-\dfrac{1}{16n^4\pi^4} & (m=n),\\[2ex] \dfrac{1}{\pi^4(m-n)^4}-\dfrac{1}{\pi^4(m+n)^4} & (m\neq n).\end{cases} \quad (3.3)$$

In (3.1) we now substitute the asymptotic expansions

$$x_n(t;\omega)=\eta_n e^{\phi(t)/\omega}\{F_n(t)+\omega G_n(t)+\omega^2 H_n(t)+\ldots\} \quad (3.4)$$

(where the η_n are constants representing initial values of the x_n) and equate to zero coefficients of like powers of ω. The systems corresponding to ω^0, ω and ω^2 are found to be, respectively,

$$F_n \cdot Z_n = 0, \tag{3.5}$$

$$G_n \cdot Z_n = -[2\phi' F_n' + (1+\sigma)\lambda_n F_n' + \phi'' F_n] + \frac{\epsilon R a^2 \sigma}{\lambda_n} \sum_{m=1}^{\infty} F_m q_{mn}^{(1)} \sin t, \tag{3.6}$$

$$H_n \cdot Z_n = -[2\phi' G_n' + (1+\sigma)\lambda_n G_n' + \phi'' G_n] - F_n'' + \frac{\epsilon R a^2 \sigma}{\lambda_n} \sum_{m=1}^{\infty} [G_m q_{mn}^{(1)} \sin t + F_m q_{mn}^{(2)} \cos t], \tag{3.7}$$

where
$$Z_n \equiv \phi'^2 + (1+\sigma)\lambda_n \phi' + \frac{\sigma}{\lambda_n}[\lambda_n^3 - R a^2(1+\epsilon \cos t)]. \tag{3.8}$$

To solve these equations, first consider (3.5). This shows that either $F_n = 0$ or $Z_n = 0$, and if the latter holds we have a quadratic for ϕ', with two roots for each n. However, it is easy to show that the largest growth-rate is determined by the greater of the roots when $n = 1$. In particular, when this mode is marginally stable, all others are damped. Hence it is sufficient to take

$$Z_1 = 0, \quad F_2 = F_3 = \ldots = 0, \tag{3.9}$$

and then we have
$$\phi' = -\tfrac{1}{2}(1+\sigma)\lambda_1 + \tfrac{1}{2}(A + B\cos t)^{\frac{1}{2}}, \tag{3.10}$$

where
$$A = (1-\sigma)^2 \lambda_1^2 + \frac{4 R a^2 \sigma}{\lambda_1} \quad \text{and} \quad B = \frac{4 \epsilon R a^2 \sigma}{\lambda_1}. \tag{3.11}$$

Next consider the system (3.6), treating separately the equations when $n = 1$ and when $n \neq 1$. When $n = 1$, $Z_1 = 0$, and so, using (3.9)–(3.11), we obtain an equation for F_1 which, after a little algebra, is found to be

$$\frac{d}{dt}(\log F_1) = \tfrac{1}{4} B \left\{ \frac{\sin t}{A + B\cos t} + \frac{q_{11}^{(1)} \sin t}{(A + B\cos t)^{\frac{1}{2}}} \right\}. \tag{3.12}$$

On the other hand, when $n \neq 1$ and $Z_n \neq 0$, equations (3.6) are algebraic expressions for the functions G_n, $n > 1$. We find

$$G_n = \tfrac{1}{4} B \frac{\lambda_1 q_{1n}^{(1)} \sin t}{\lambda_n Z_n} F_1 \quad (n = 2, 3, \ldots). \tag{3.13}$$

Next we determine G_1. This is given from the first member of (3.7), i.e. with $n = 1$. It is easily shown that

$$\frac{d}{dt}[G_1/F_1] = \frac{1}{(A + B\cos t)^{\frac{1}{2}}} \left\{ -\frac{F_1''}{F_1} + \tfrac{1}{4} B q_{11}^{(2)} \cos t + \frac{B^2 \sin^2 t}{16} \sum_{m=2}^{\infty} \frac{\lambda_1 q_{1m}^{(1)2}}{\lambda_m Z_m} \right\}. \tag{3.14}$$

This solution procedure can be continued indefinitely, and provides an asymptotic approximation for the fastest-growing mode, namely

$$x_1 \sim \eta_1 e^{\phi/\omega}[F_1 + \omega G_1 + \ldots], \tag{3.15}$$

where ϕ, F_1 and G_1 are given by (3.10), (3.12) and (3.14) respectively. The solution is subject to a restriction, always present in the WKB method, that the coefficient of the dependent variable in the normalized form of the differential equation should be free from zeros. In the present case it is immediately clear from inspection of (3.12) and (3.14) that we require

$$A + B \cos t \neq 0 \tag{3.16}$$

in the interval of interest, which is $(0, 2\pi)$. Hence condition (3.16) is equivalent to

$$A > B. \tag{3.17}$$

This places a limitation on the possible range of ϵ. As (3.11) shows, if $\sigma = 1$ we are restricted to $\epsilon < 1$; but if $\sigma \neq 1$, the value of ϵ may be greater than unity, so long as (3.17) is satisfied.

It is of course possible to obtain an asymptotic solution even when (3.17) does not hold. Such a solution will be in terms of Airy functions, as is usual in the WKB method when a zero is present. Because of the additional complications, we shall not pursue this case in the present paper, though the general discussion which appears below is equally applicable.

The expression (3.15) is merely a representation of a solution to the differential equations (3.1), and no conditions of stability have yet been introduced. It is well known (cf. Conrad & Criminale 1965; Rosenblat 1968) that alternative stability criteria are possible when the equilibrium state is time-periodic. In the next two sections we examine the consequences of applying two different criteria to the solution (3.15).

4. Periodicity criterion

This is a natural criterion to use in discussing the stability of time-dependent systems, and is the one used by Venezian (1969) in the present configuration. The object is to determine the value R_p, say, of the Rayleigh number R for which the disturbance x_1 is periodic with period 2π. Since x_1 is the least damped of the modes, and since (3.1) is a system with periodic coefficients to which standard Floquet theory is applicable, we are assured that all disturbances will decay when $R < R_p$, while at least x_1 will grow when $R > R_p$.

We can write this condition on x_1 in the form

$$\log \left| \frac{x_1(2\pi)}{x_1(0)} \right| = 0, \tag{4.1}$$

and, on applying this to (3.15), we obtain to order ω^2,

$$\phi(2\pi) - \phi(0) + \omega^2 \left[\frac{G_1}{F_1}(2\pi) - \frac{G_1}{F_1}(0) \right] = 0. \tag{4.2}$$

Here ϕ and G_1/F_1 are given by (3.10) and (3.14) respectively. Equation (4.2) contains no term of order ω because its coefficient, $\log F_1$, is identically periodic.

In equation (4.2) we substitute the expansion

$$R = R_p = R_p^{(0)} + \omega^2 R_p^{(1)} + \dots \tag{4.3}$$

corresponding to which we put

$$A = A_0 + \omega^2 A_1 + \ldots, \quad B = B_0 + \omega^2 B_1 + \ldots, \tag{4.4}$$

where, by (3.11)

$$\left. \begin{aligned} A_0 &= (1-\sigma)^2 \lambda_1^2 + \frac{4a^2\sigma R_p^{(0)}}{\lambda_1}, \quad B_0 = \epsilon \frac{4a^2\sigma R_p^{(0)}}{\lambda_1}, \\ A_1 &= \frac{4a^2\sigma R_p^{(1)}}{\lambda_1}, \quad B_1 = \epsilon \frac{4a^2\sigma R_p^{(1)}}{\lambda_1} = \epsilon A_1. \end{aligned} \right\} \tag{4.5}$$

We now insert these forms in the above-mentioned expressions for ϕ and G_1/F_1 in (4.2). When coefficients of like powers of ω are equated in the latter, we obtain, to order ω^2,

$$-\pi(1+\sigma)\lambda_1 + \frac{1}{2}\int_0^{2\pi} (A_0 + B_0 \cos s)^{\frac{1}{2}}\, ds = 0 \tag{4.6}$$

and

$$\frac{1}{4}\int_0^{2\pi} \frac{A_1 + B_1 \cos s}{(A_0 + B_0 \cos s)^{\frac{1}{2}}}\, ds + \left[\frac{G_1}{F_1}(2\pi) - \frac{G_1}{F_1}(0)\right]_{\omega=0} = 0. \tag{4.7}$$

Equations (4.6) and (4.7) serve to determine $R_p^{(0)}$ and $R_p^{(1)}$ respectively in terms of the other parameters.

The integral in (4.6) is an elliptic integral, and hence (4.6) becomes

$$\pi(1+\sigma)\lambda_1 = 2(A_0 + B_0)^{\frac{1}{2}} E, \tag{4.8}$$

where E is the complete elliptic integral of the second kind, dependent on a parameter k defined by

$$k^2 = 2B_0/(A_0 + B_0) \tag{4.9}$$

(cf. Byrd & Friedman 1954). Since $E = E(k)$ is tabulated, the value of $R_p^{(0)}$ is easily obtained numerically.

Equations (4.5), (4.8) and (4.9) show that $R_p^{(0)}$ is a function of the three parameters a^2, σ and ϵ, and its variation with each of these is now considered in turn. If we write

$$R_c = \lambda_1^3/a^2, \tag{4.10}$$

which is the critical Rayleigh number of the classical Bénard problem, whose critical wave-number is given by

$$a^2 = \tfrac{1}{2}\pi^2, \tag{4.11}$$

we can easily show that equations (4.8) and (4.9) can be expressed in the form

$$\pi^2(1+\sigma)^2 = 4[(1-\sigma)^2 + 4\sigma(1+\epsilon) R_p^{(0)}/R_c] E^2 \tag{4.12}$$

with

$$k^2 = \frac{8\epsilon\sigma R_p^{(0)}/R_c}{(1-\sigma)^2 + 4\sigma(1+\epsilon) R_p^{(0)}/R_c}. \tag{4.13}$$

Since these two expressions do not contain a explicitly, it follows that $R_p^{(0)}$ and R_c have the same wave-number dependence in the sense that the critical value of a for both of them is given by (4.11).

In the subsequent discussion this critical wave-number will be understood, so that R_c in (4.12) and (4.13) has the value

$$R_c = 27\pi^4/4. \tag{4.14}$$

We next turn to the variation of $R_p^{(0)}$ with σ. It is easy to verify analytically, from (4.12) and (4.13), that the quantity $R_p^{(0)}/R_c$ is a maximum when $\sigma = 1$ (for arbitrary, fixed ϵ). This means that in the limit $\omega \to 0$ the enhancement of the critical Rayleigh number is greatest when $\sigma = 1$. The actual variation of $R_p^{(0)}/R_c$ over a range of values of σ has been calculated numerically, and is illustrated in figure 1 for a typical value of ϵ, namely, $\epsilon = 1$.

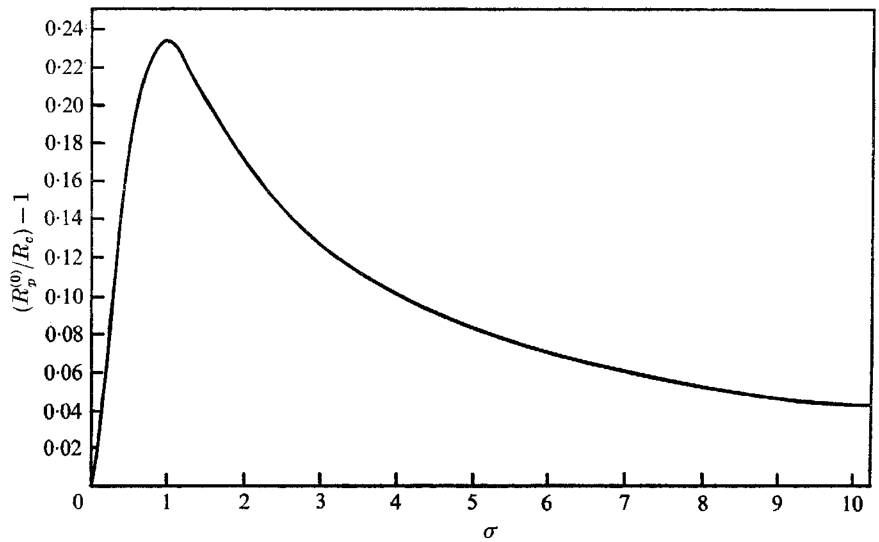

FIGURE 1. Periodicity criterion: $R_p^{(0)}/R_c$ as a function of σ, with $\epsilon = 1$.

Finally, we consider the variation with ϵ. For the purposes of illustration, it is sufficient to take the peak modulation value $\sigma = 1$. In this case (4.12) and (4.13) reduce to the simple forms

$$\frac{R_p^{(0)}}{R_c} = \frac{\pi^2}{4(1+\epsilon)E^2} \quad \text{and} \quad k^2 = \frac{2\epsilon}{1+\epsilon}. \tag{4.15}$$

These are now evaluated numerically, and figure 2 shows the behaviour of $R_p^{(0)}/R_c$ in the range $0 \leqslant \epsilon < 1$.

We may mention in passing that an alternative simplification of (4.12) is possible when $\epsilon \ll 1$. Using the series expansion for the elliptic integral, we obtain

$$\frac{R_p^{(0)}}{R_c} = 1+\epsilon^2 \frac{2\sigma}{(1+\sigma)^2}+O(\epsilon^4), \tag{4.16}$$

which is identical with the result obtained by Venezian (1969).

It has been noted earlier that the correction $R_p^{(1)}$ to the critical Rayleigh number can be determined from (4.7). The evaluation of the various terms in this equation can be accomplished in terms of elliptic functions and integrals, and is straightforward, though laborious. We shall not give any details here, but rather state the two main features which emerge from the calculations: (i) For the entire ranges of ϵ and σ under consideration, $R_p^{(1)}$ is found to be *negative*. This means that the enhancement is a maximum when $\omega = 0$, which, as has been mentioned in § 1, does not agree with the observations of Donnelly (1964). (ii) The incremental ratio $R_p^{(1)}/R_p^{(0)}$ is very small, generally $O(10^{-2})$. This is why we do not feel it worthwhile to pursue the details here; when $\epsilon \ll 1$, they are given by Venezian (1969).

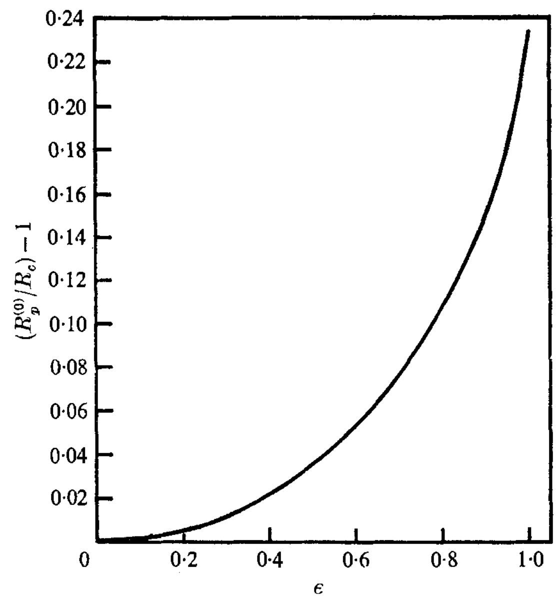

FIGURE 2. Periodicity criterion: $R_p^{(0)}/R_c$ as a function of ϵ, with $\sigma = 1$.

It is relevant now to consider whether periodicity is a suitable criterion of stability in the present situation. When $R < R_p$ the solution obtained above is certainly *quasi-asymptotically stable*. That is, any infinitesimal disturbance, whose initial value η_1 is less than some number η, tends to zero as $t \to \infty$. It is, of course, not possible to determine η; it corresponds to the maximum amplitude of those disturbances for which linearization is valid.

It can also be said that disturbances are *stable* (in the Lyapunov sense) for $R \leqslant R_p$, in that we can find a number $\tilde{\eta}$ such that all disturbances with $\eta_1 < \tilde{\eta}$ can be made to remain within prescribed bounds at all times t. This property, however, is subject to the following qualification. The form of the solution shows that when ω is small and R is close to R_p the quantity $e^{\phi/\omega}$ oscillates between very large and very small values. In the limit $\omega \to 0$ this oscillation tends to infinity. It follows that if $|x_1|$ is to have a given finite bound, its initial value must be chosen sufficiently small. In particular, $\tilde{\eta}$ will need to depend on ω, with a variation of the form $e^{-1/\omega}$, and cannot be prescribed independently of ω. Thus, although we can achieve stability, we cannot ensure stability of *any* disturbance uniformly with respect to ω as $\omega \to 0$.

If, on the other hand, we consider the class of disturbances specified by η, we may have sufficiently substantial growth during some interval of time for the linear theory to break down. In the next section we suggest how it might be possible to modify the stability criterion so as to include the class η within linear theory.

5. Amplitude criterion

We now propose a criterion that infinitesimal disturbances should be stable uniformly with respect to ω. The object of this criterion is to prevent the amplitude of the least stable mode, x_1, becoming too large during *any part* of its cycle.

Let the new critical Rayleigh number be R_a. Then obviously $R_a < R_p$, and the mode x_1 will have the form of a damped oscillation. Let $(x_1)_{\min}$ and $(x_1)_{\max}$ be the values of x_1 at a successive minimum and maximum respectively, occurring at $t = \tau_-$ and $t = \tau_+$, say. Then we can express our 'amplitude criterion' by stating that the disturbance is stable if

$$\log\left[\frac{(x_1)_{\max}}{(x_1)_{\min}}\right] \leqslant M, \tag{5.1}$$

where $M = O(1)$ as $\omega \to 0$. The equality sign in (5.1) corresponds to marginal stability.

From (3.15),

$$\frac{(x_1)_{\max}}{(x_1)_{\min}} = \exp\left[(1/\omega)\{\phi(\tau_+)-\phi(\tau_-)\}\right]\left[\frac{F_1(\tau_+)}{F_1(\tau_-)}+O(\omega)\right], \tag{5.2}$$

so that (5.1) takes the form

$$\frac{1}{\omega}[\phi(\tau_+)-\phi(\tau_-)]+\log\left[\frac{F_1(\tau_+)}{F_1(\tau_-)}+O(\omega)\right] \leqslant M. \tag{5.3}$$

The times $t = \tau$ at which $x_1(t)$ is stationary are given by the zeros of $x_1'(t)$, that is, by the equation

$$\left[\frac{\phi' F_1}{\omega}+\phi' G_1+F_1'+O(\omega)\right]_{t=\tau} = 0. \tag{5.4}$$

Equations (5.3) and (5.4) have now to be solved simultaneously. To do this we assume an expansion for the critical Rayleigh number in powers of ω, which in this case turn out to be non-integral powers. We put

$$R = R_a = R_a^{(0)}+\omega^N R_a^{(1)}+\ldots, \tag{5.5}$$

where N is to be determined. Corresponding to (5.5) we write

$$A = A_0+\omega^N A_1+\ldots, \quad B = B_0+\omega^N B_1+\ldots, \tag{5.6}$$

where the A's and B's are as defined in (4.5), except that $R_p^{(0)}$, $R_p^{(1)}$ are replaced by $R_a^{(0)}$, $R_a^{(1)}$ respectively.

In addition, the consecutive stationary times τ_-, τ_+ have the following property. If $t = 0$ is taken as an arbitrary reference point, then τ_- and τ_+ coalesce to $t = 0$ when $\omega \to 0$.† This is apparent from consideration of (5.3) and (5.4) in the limit $\omega \to 0$:

$$\phi(\tau_+) = \phi(\tau_-), \quad \phi'|_{t=\tau} = 0. \tag{5.7}$$

† The zeros must be obtained from an analysis of (3.10), which shows that their locations depend on ω.

Consequently we may write

$$\tau = 0+\omega^{\frac{1}{2}N}\tau_1+\ldots. \tag{5.8}$$

(We have circumvented some algebra here by assuming a power of ω which is justified *a posteriori*.)

Now substitute (5.5), (5.6) and (5.8) into (5.3) and (5.4), and equate coefficients of like powers of ω. First consider (5.4). The two leading equations, corresponding to the powers ω^0 and ω^N respectively, are found to be

$$-\tfrac{1}{2}(1+\sigma)\,\lambda_1+\tfrac{1}{2}(A_0+B_0)^{\frac{1}{2}} = 0 \tag{5.9}$$

and

$$A_1+B_1-\tfrac{1}{2}B_0\,\tau_1^2 = 0. \tag{5.10}$$

Equation (5.9) gives immediately

$$R_a^{(0)} = R_c/(1+\epsilon). \tag{5.11}$$

This result, which could have been anticipated, is the quasi-steady formula: it states that the temperature gradient is not unstable at any point of the cycle.

Using (4.5) in (5.10), we have

$$\tau_1 = \pm\left[\frac{2A_1(1+\epsilon)}{B_0}\right]^{\frac{1}{2}} = \pm\left[\frac{2(1+\epsilon)\,R_a^{(1)}}{\epsilon R_a^{(0)}}\right]^{\frac{1}{2}}. \tag{5.12}$$

Now consider (5.3). This can be written

$$\int_{\tau_-}^{\tau_+}\phi'(s)\,ds+O(\omega^2) = \omega M \tag{5.13}$$

for marginal stability. From (3.10) and the preceding results, we easily show that

$$\int_{\tau_-}^{\tau_+}\phi'(s)\,ds = \omega^{\frac{3}{2}N}\left(\frac{R_a^{(1)}}{R_a^{(0)}}\right)^{\frac{3}{2}}.\frac{2^{\frac{5}{2}}}{3}\frac{\sigma\lambda_1}{1+\sigma}\left(\frac{1+\epsilon}{\epsilon}\right)^{\frac{1}{2}}. \tag{5.14}$$

Substitution of this into (5.13) gives

$$N = \tfrac{2}{3} \tag{5.15}$$

and

$$\frac{R_a^{(1)}}{R_a^{(0)}} = \frac{1}{2}\left[\frac{3(1+\sigma)\,M}{2\sigma\lambda_1}\right]^{\frac{2}{3}}\left[\frac{\epsilon}{1+\epsilon}\right]^{\frac{1}{3}}. \tag{5.16}$$

Finally, we substitute (5.11), (5.15) and (5.16) into (5.5). This now becomes

$$R_a = \frac{\lambda_1^3}{(1+\epsilon)\,a^2}[1+\omega^{\frac{2}{3}}.\mu+\ldots], \tag{5.17}$$

where

$$\mu = \frac{1}{2}\left[\frac{3(1+\sigma)\,M}{2\sigma\lambda_1}\right]^{\frac{2}{3}}\left[\frac{\epsilon}{1+\epsilon}\right]^{\frac{1}{3}}. \tag{5.18}$$

Equation (5.17) gives the value of the critical Rayleigh number according to the amplitude criterion stated above. This value is naturally indeterminate to the extent that the constant M is not specified within the criterion. The qualitative trend, however, is clear: the Rayleigh number increases with increasing ω, commencing with the quasi-steady value at $\omega = 0$.

It is also apparent from (5.17) that the critical wave-number is in this case subject to a second-order modification, of order $\omega^{\frac{2}{3}}$. The exact form again depends on the undetermined constant M.

The nature of the solution (5.17) is illustrated in figure 3. For definiteness, we have chosen $M = 2{\cdot}25$, which corresponds to a ten-fold amplification of the disturbance. Figure 3 shows the variation of R_a/R_c with ω, in the range $0 < \omega < 1$, for several values of ϵ and with $\sigma = 1$.

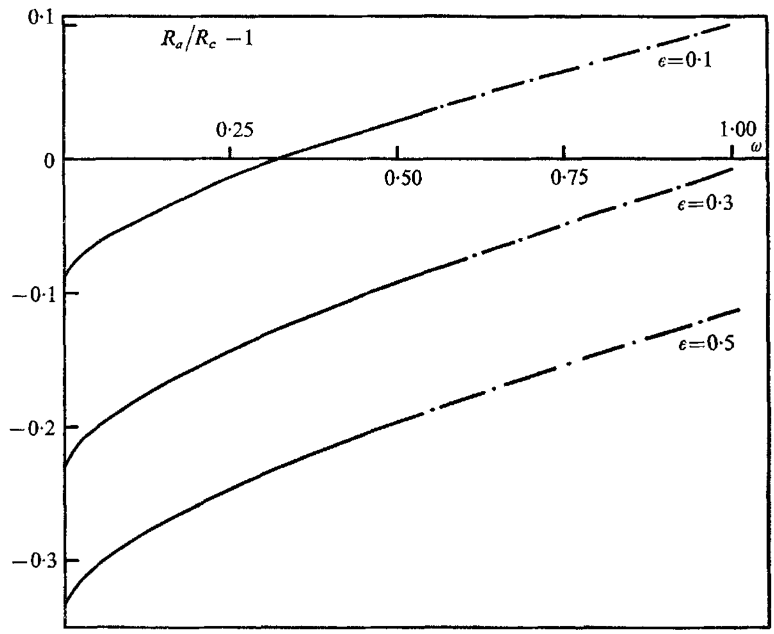

FIGURE 3. Amplitude criterion: R_a/R_c as a function of ω, with $\sigma = 1$, $M = 2{\cdot}25$, $a^2 = \frac{1}{2}\pi^2$, and for values $\epsilon = 0{\cdot}1,\ 0{\cdot}3,\ 0{\cdot}5$.

6. Conclusions

The discussion of the preceding two sections has been concerned with the interpretation of the asymptotic solution for small ω, and can be summarized as follows. For a certain class of disturbances (those for which linearization is valid and whose initial values are independent of frequency) the periodicity criterion is a *sufficient condition for instability*, in that the inequality

$$R > R_p \tag{6.1}$$

ensures asymptotic growth of the disturbances according to linear theory.

On the other hand, the condition

$$R < R_a \tag{6.2}$$

can be regarded as a *sufficient condition for stability* (on linear theory) for the *same* class of disturbances, in the sense that their magnitude at any instant can be *a priori* restricted.

In the case $$R_a < R < R_p \tag{6.3}$$
our solutions predict that *these* disturbances decay as $t \to \infty$, but have extreme oscillations at finite t for $\omega \to 0$. It is not possible to predict the actual behaviour of such disturbances without a non-linear analysis. That is, we cannot say whether non-linear effects tend to stabilize, by decreasing the amplitude of the oscillations, or to destabilize, by reinforcing them.

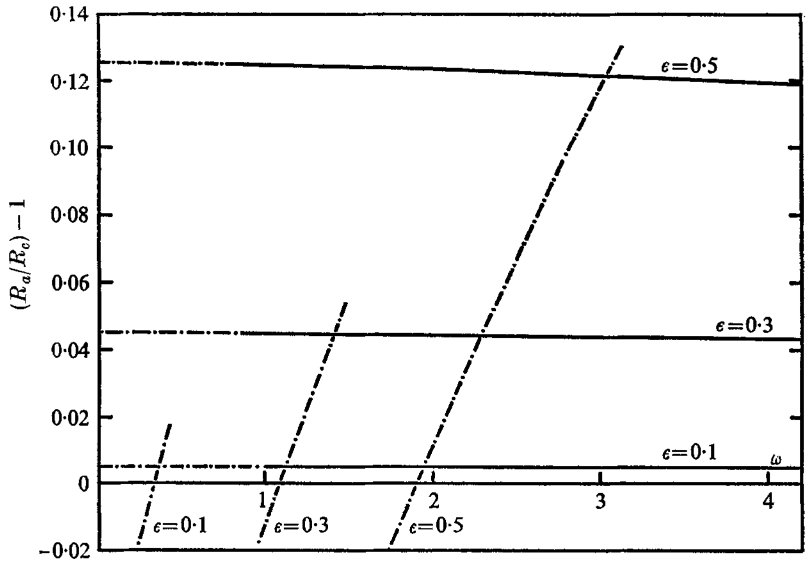

FIGURE 4. Variation of R_a/R_c and R_p/R_c with ω (extrapolated), with $\sigma = 1$, $M = 2·25$, $a^2 = \frac{1}{2}\pi^2$, and for values $\epsilon = 0·1$, $0·3$, $0·5$.

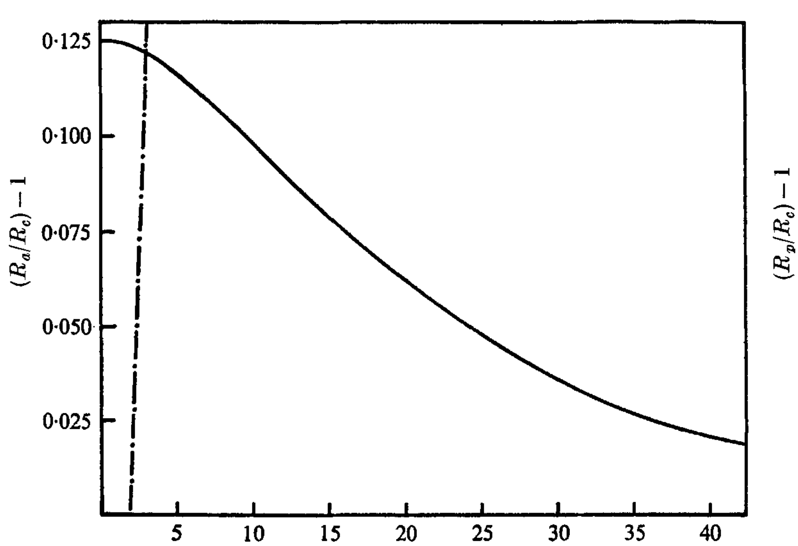

FIGURE 5. Variation of R_a/R_c and R_p/R_c with ω (extrapolated) with $\sigma = 1$, $M = 2·25$, $a^2 = \frac{1}{2}\pi^2$, $\epsilon = 0·5$.

Finally, when $R < R_p$ there is a class of disturbances which not only decay as $t \to \infty$ but also have finite bounds on their variation. This class, however, contains only disturbances with frequency-dependent initial values, and becomes vanishingly small as $\omega \to 0$.

It follows that the question of which criterion is relevant to an experimental result can only be answered in terms of which disturbances are observed, and this, of course, is not known. In this sense, therefore, the problem remains open.

In figures 4 and 5 we have represented R_p and R_a on the same diagram, using R_a at the lowest values of ω and R_p at slightly larger values. Figure 4 illustrates these quantities for various values of ϵ, while figure 5 depicts them, on a different scale, at $\epsilon = 0{\cdot}5$. In both figures the solid lines refer to R_p and the broken line to R_a.

Although these graphs are somewhat speculative, and can only be qualitative, because of the indeterminacy in R_a, it is interesting to note that the combination of R_p and R_a described above yields a value of ω at which the enhancement is a maximum. With $M = 2{\cdot}25$ (see § 5 above), this value turns out to be $\omega = O(1)$, which is in broad agreement with Donnelly's experimental results.

It may therefore be inferred that at low frequencies the large oscillations implied in the periodicity criterion do in fact lead to instability, at any rate in the particular experiment under consideration.

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